Resource Material Prototype

Mathematics GLE Resource Materials

Definitions and Examples for Grades K – 8

Functions and Algebra

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New Hampshire Department of Education
Rhode Island Department of Education
Vermont Department of Education

Resource Material Prototype

Overview

The purpose of these resource materials is to provide K – 8 educators with explanations and examples that facilitate understanding of the mathematics in the Grade-Level Expectations (GLEs)*. These resource materials are organized by content strand and by GLE “**stems**.” The definitions and explanations within each section are not alphabetized, but are organized to parallel the introduction of new concepts and skills within each GLE “**stem**” across grades. *The materials contained in this document focus on the Functions and Algebra strand.*

Stem

Grade 3	Grade 4
M(F&A)–3–1 **Identifies and extends to specific cases a variety of patterns** (linear and non-numeric) represented in models, tables, or sequences by extending the pattern to the next one, two, or three elements, or finding missing elements.	M(F&A)–4–1 **Identifies and extends to specific cases a variety of patterns** (linear and nonlinear) represented in models, tables or sequences; and writes a rule in words or[sc] symbols to find the next case.

To facilitate access, each definition is coded (e.g., ***F&A–32***, the 32nd definition for the Function and Algebra Strand). There are multiple tables of contents: 1) An overall table of contents on page 3 contains all the terms or phrases defined for the content strand in alphabetical order; and 2) A table of contents for each section in alphabetical order. In addition, if there is a word or phrase that you are unclear about within a definition, check the overall table of contents to determine if the word or phrase is defined in another location in this document.

Section #	**Grade Level Expectation “Stem”**	**NECAP GLE Stem Codes (Vermont codes)** X represents grades K–8	**Page Numbers**
Section 1	Identifies, extends and generalizes patterns	NECAP: M(F&A) – X – 1 (Vermont GLE MX: 19)	4 – 17
Section 2	Linear and non-linear relationships and functions	NECAP: M(F&A) – X – 2 (Vermont GLE MX: 20)	18 – 44
Section 3	Algebraic expressions	NECAP: M(F&A) – X – 3 (Vermont GLE MX: 21)	45 – 48
Section 4	Demonstrates understanding of equality	NECAP: M(F&A) – X – 4 (Vermont GLE MX: 22)	49 – 55

Note: Examples are provided throughout this document to illustrate definitions or phrases used in the mathematics GLEs. However, the kinds of questions students might be asked in instruction or on the NECAP assessment about the mathematics being illustrated are NOT necessarily limited to the *specific* examples given. It is the intent, over time, to add released NECAP items to expand the set of examples.

Please send comments, corrections, or suggestions to mathsurvey@gmavt.net.

* Grade Level Expectations are called Grade Expectations (GEs) in Vermont.

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Section 1: Identifies, extends and generalizes patterns

NECAP: M(F&A) – X – 1
Vermont: MX: 19

	Page #	Definition #
Expresses generalization or rule using words or symbols	15	*F&A – 7*
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Generalizes a pattern to find a specific case	16	*F&A – 8*
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Section 1: Identifies, extends and generalizes patterns

F&A – 1 **Pattern:** A pattern is a sequence (an ordered set of objects/numbers – see ***F&A – 6***) that can be described by a rule.

Rules may be written for all elements of a set (Example 1.1) or for some elements of a set (Example 1.2). Patterns can be linear or nonlinear and can be numeric or non-numeric (e.g., objects, colors, or letters).

$$\overbrace{\{2, 4, 6, 8, 10,\ldots\}}^{\text{Elements in the set}}$$

Related terminology
- A **set** is a well defined collection of objects.
- An **element/term** is an object of a set.
- A **case** or **term number** is the position of the element in the set.

The elements in this set are 2, 4, 6, 8, 10,....
- 2 is the first case (or first term) in the set.
- 6 is the third case (or third term) in the set.

- A set can have elements that are numbers, geometric objects, or other objects such as colors, sounds, words, or people. In general, the elements in a set do not have to be ordered.

- Patterns can be found in various mathematical representations including tables, graphs, models, and in problem situations.

- A rule that applies to all the elements of a set is a generalized rule which is prompted by “find the rule for the n^{th} term (any term).”

Every generalization is a rule, but not every rule is a generalization. (e.g., Example 2.1 is a recursive rule that is not a generalization. Example 3.1 shows both a recursive rule that is NOT a generalization and a non-recursive rule that is a generalization.)

- A set is a sequence if the elements are ordered. Whenever one is asked to “find a specific case” or to “find the rule for the n^{th} case” it is assumed that the elements in the set are ordered and, therefore, the set is a sequence. Furthermore, to indicate an ordered set, set brackets ({ , }) are typically omitted (e.g., 2, 4, 6, 8, 10,…).

When students investigate sets of objects or numbers they sometimes identify a rule that applies to all elements of the set (the n^{th} term), and they sometimes identify rules that apply for just some elements of the set.

(Definition *F&A – 1* continued on following page)

Example 1.1 – A rule that applies to all elements of a set (n^{th} term):

2, 4, 6, 8, 10,… is the sequence (ordered set) of even counting numbers. The general rule for finding the n^{th} term (any term) in this sequence is multiply the "case" number by 2.

n^{th} term of sequence 2, 4, 6, 8, 10, …

Position (Case #)	1	2	3	4	5	6	7	…	n
Element (Term)	2	4	6	8	10	12	14	…	$2n$

An ellipsis (…) is used to show that an element or elements in a pattern are missing.

Example 1.2 – A rule that applies to some elements of a set: Discounting 1 as a factor, some elements of the sequence (ordered set) 2, 4, 6, 8, 10, 12, 14, 16,… have **only even** number factors (i.e., 2, 4, 8, 16,...) and other elements of the set have **even and odd** number factors (i.e., 6, 10, 12, 14,…).

Examining subsets of the original set is important mathematics and should be a regular part of instruction (Example 1.2). However, unless otherwise stated, NECAP items will require students to find the rule that allows for determining the next case, a specific case, missing elements, or the n^{th} case (i.e., a rule that applies to all elements of a set).

F&A – 2 **Recursive rules and non-recursive rules:** A recursive rule is a relation between successive elements that specifies how to calculate the value of a specific element based on previous elements (Example 2.1). Non-recursive rules are based on the relationship between the case number and the elements of a set (Example 2.2).

Example 2.1 – A recursive rule: In this situation, the recursive rule is to add 3 to the value of the previous element.

Case #	Element
1	4
2	7
3	10
4	13
5	**?**

+3

+3

+3

+3

The difference between successive elements is 3.

? = 13 + 3
? = 16

Example 2.2 – A non-recursive rule: In this situation, a non-recursive rule is to add 1 to 3 times the case number. Unlike a recursive rule, a non-recursive rule can be applied to any case whether it is the 5th case or the 1000th case **efficiently**.

Case #	Element
1	$(3 \times 1) + 1 \rightarrow$ 4
2	$(3 \times 2) + 1 \rightarrow$ 7
3	$(3 \times 3) + 1 \rightarrow$ 10
4	$(3 \times 4) + 1 \rightarrow$ 13
5	?

? = (3 × 5) + 1
? = 16

F&A – 3 **Numeric patterns:** Numeric patterns can be linear or nonlinear.

Linear patterns are represented by constant rates of change and form straight lines when graphed.

Example 3.1 – A linear pattern: The sequence 3, 5, 7, 9, 11, 13,… is a linear pattern represented in the table and graph below. Notice how the points, if joined, would form a straight line. This is the key feature of a linear pattern.

If the points were joined the pattern would form a straight line.

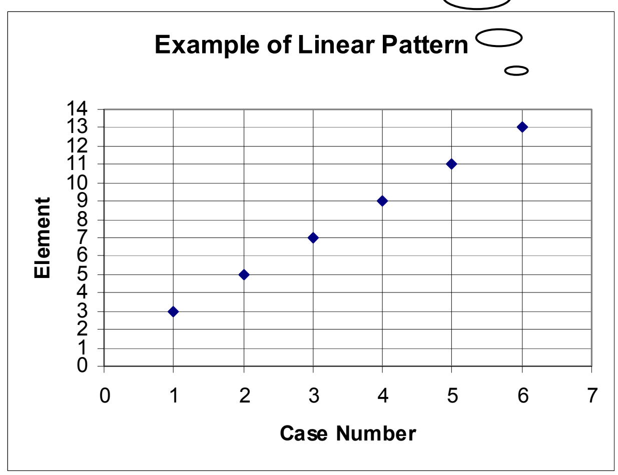
Recursive rule

Case #	Element
1	3
2	5
3	7
4	9
5	11
6	13
⋮	⋮
n	?

+ 2

(2 · Case #) + 1 = Value of Term
or
(2 · *n*) + 1 = Value of Term

Non-recursive rule



Recursive rule: Following the first term, each term is two more than the previous term.
Non-recursive rule: For any case of the sequence, the value of the corresponding element is two times the case number + 1.

(Definition *F&A – 3* continued on following page)

Nonlinear patterns are represented by varying rates of change and do not form straight lines when graphed.

Example 3.2 – A nonlinear pattern: 1, 4, 9, 16,…

Case #	1	2	3	4	5
Element/Term	1	4	9	16	?

Non-recursive rule
(Case #)^2 or ? = 5^2 = 25

3 5 7 9

$\boxed{7 + 2 = 9}$

? = 9 + 16 = 25

Recursive rule

Recursive rule: The recursive rule for this pattern is based on differences between each successive term forming a pattern of 3, 5, 7,. . . . Therefore, each term (after Term 2) is the previous term plus two more than the difference between the previous two terms. The next term (for Case 5) would be 9 more than the term for Case 4 following this recursive rule.
Non-recursive rule: For any case of the set, the value of the corresponding element is the case number squared (n^2).

Common nonlinear patterns include patterns such as square numbers, cube numbers, and growing patterns.

Example 3.3 – A growing pattern: 1, 2, 4, 7, 11, 16,…

Recursive rule: The *n*th term (for $n > 2$) is the previous term plus one more than the difference between the previous two successive terms.

F&A – 4 **Non-numeric patterns:** Non-numeric patterns are patterns using objects, sounds, colors, visual/geometric models, or other symbols.

Example 4.1 – Non-numeric Repeating patterns:

AAB Pattern: $\underbrace{\text{red, red, blue,}}_{A\ A\ B}\ \underbrace{\text{red, red, blue,}}_{A\ A\ B}\ \underbrace{\text{red, red, blue,}}_{A\ A\ B}$. . .

ABB Pattern (when clapping): $\underbrace{\text{soft, loud, loud,}}_{A\ B\ B}\ \underbrace{\text{soft, loud, loud,}}_{A\ B\ B}\ \underbrace{\text{soft, loud, loud,}}_{A\ B\ B}$ . . .

ABC Pattern: $\underbrace{\bullet\ \blacktriangle\ \blacksquare}_{A\ B\ C}\ \underbrace{\bullet\ \blacktriangle\ \blacksquare}_{A\ B\ C}\ \underbrace{\bullet\ \blacktriangle\ \blacksquare}_{A\ B\ C}$, . . .

Example 4.2 – Non-numeric Non-Repeating patterns: Some non-repeating numeric patterns can be thought of as non-numeric patterns. For example, if Example 5.1 (page 11) asked students to determine the sixth figure in the pattern, it may be determined by finding the total number of blocks (numeric reasoning) or by adding another column of blocks to find the fifth figure and then another column of blocks to find the sixth figure (non-numeric reasoning).

F&A – 5 **Extend a pattern:** When a pattern is extended, the set of elements is expanded to a larger set, and the rule is extended across the larger set.

Example 5.1 – Extending a pattern in a visual/geometric pattern.

If the pattern continues, how many blocks will be in the 6th figure?

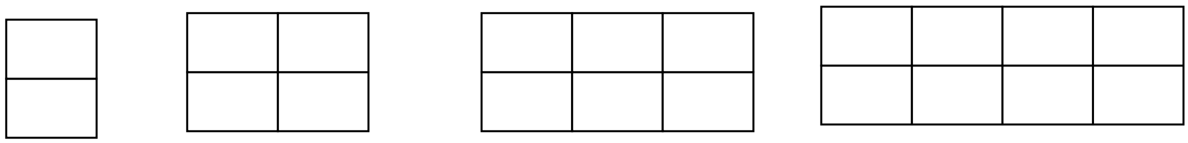


Figure 1 Figure 2 Figure 3 Figure 4

Key


represents 1 block

Figure Number	Number of Blocks
1	2
2	4
3	6
4	8
5	10
6	?

The question "How many blocks would be in Figure 6?" requires that the patterns be extended to a specific case (Figure 6).

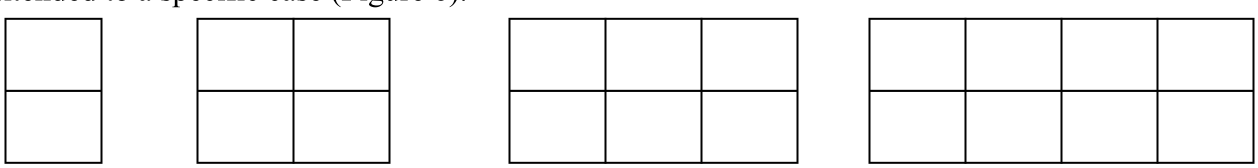


Figure 1 Figure 2 Figure 3 Figure 4

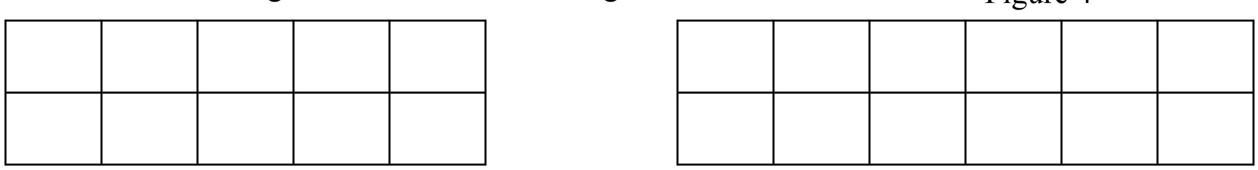


Figure 5 Figure 6

Answer: In Figure 6, there are 12 blocks.
Recursive rule: Following the first term, each figure has two more blocks than the previous figure.
Non-recursive rule: For any case of the set, the value of the corresponding element is two times the figure (case) number.

(Definition *F&A – 5* continued on following page)

Example 5.2 – Extending a pattern in a table to next or other cases: What is the value of the element of the 5th case, the 25th case, and the 100th case?

Case #	Element/Term
1	2
2	4
3	6
4	8
5	
6	
7	
⋮	⋮
25	
⋮	⋮
100	

5th case

100th case

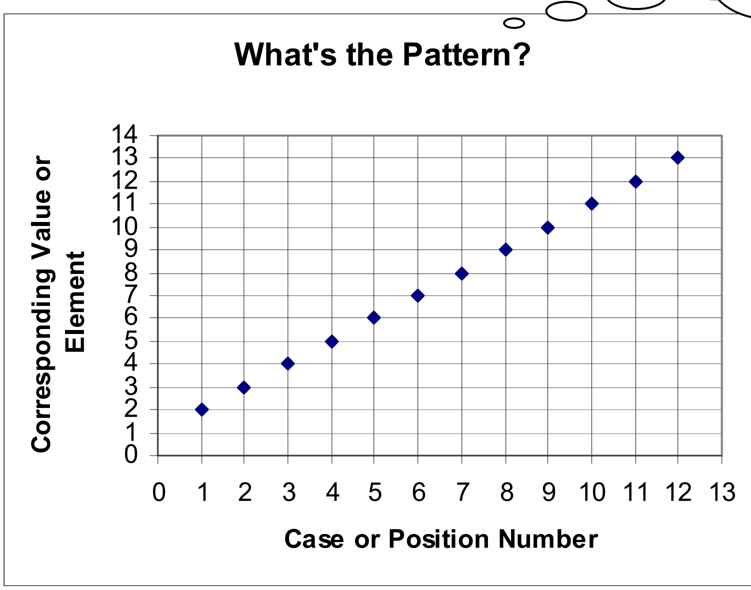
10

50

200

Example 5.3 – Extending a pattern in a graph to a rule:

Add 1 to the case number to obtain the corresponding value.

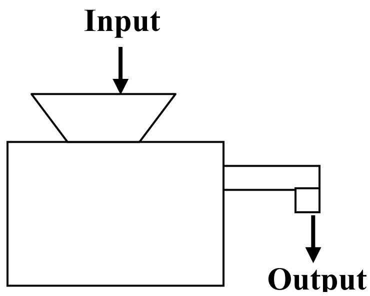


Answer: $c = n + 1$, where c is element corresponding to the nth position.
(**Note:** The rule given is a generalization of the pattern to the nth case.)

(Definition *F&A – 5* continued on following page)

Example 5.4 – Extending a pattern represented in a model to other cases and to the *n*th case:

Greg has been asked to find the rule for the function machine shown below. He inputs four numbers as a trial to see how the machine works. The results of the trial are shown in the table below.



Input	Output
3	13
7	29
2	9
10	41

Item adapted from NH Grade 6 2002 NHEIAP

a. If Greg inputs the number 5, what will the **output** be? (Answer: 21)
b. If the input is 12, what is the **output**? (Answer: 49)
c. What **input** would produce an output of 61? (Answer: 15)
d. If the input is the variable *n*, write the rule that generates the **output**. (Answer: Output $= 4n + 1$)

Example 5.5 – Pattern represented in a problem situation extended to other cases:

Tom collected seashells during his summer vacation. The table below shows how many shells are in his collection during the first four days of his vacation.

Tom's Seashell Collection

Day	Number of Seashells in Tom's Collection
1	5
2	8
3	11
4	14
⋮	⋮
21	

Item adapted from Vermont Draft Test Specifications Version 42.0

If this pattern continues, how many shells will be in Tom's collection on day 21 of his vacation?

Answer: Tom will have 65 seashells in his collection by Day 21.
Recursive rule: Following the first day, the number of seashells is three more than the previous day. If this rule is applied to each successive day until day 21, there are 65 seashells in Tom's collection.
Non-recursive rule: To find the number of seashells on any day, multiply the day number by 3 and then add 2 (i.e., $3n + 2$, where n represents the day number). Therefore, the number of seashells in Tom's collection on Day 21 is $(3 \cdot 21) + 2 = 65$.

F&A – 6 **Sequence:** A sequence is an ordered set of objects/numbers and can be linear or nonlinear.

A note about sequences and patterns…

NECAP items focus on sequences that can be described by rules (see *F&A – 1* Pattern). The words sequence and pattern will be used interchangeably. Typically, sequence is used at the upper elementary grades while pattern is used at the lower elementary grades.

Example 6.1 – Linear sequence:

1, 3, 5, 7,… (**Recursive rule:** Following the first term, each term is two more than the previous term.)

Example 6.2 – Nonlinear sequences:

2, 4, 8, 16,… (**Recursive rule:** Following the first term, each term is double the previous term.)

The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34,… (**Recursive rule:** Following the first two terms, each term is the sum of the two previous terms.)

1, 5, 4, 8, 7, 11, 10, 14, 13,… (**Recursive rule:** In turn, add 4 and then subtract 1)

Example 6.3: Write a rule using words or symbols for the n^{th} term in the following sequence. Justify your rule.

Position in Sequence	1	2	3	4	…	n
Term	5	9	13	17	…	?

Non-recursive explicit equation: $t = 4n + 1$, where t is the term corresponding to the nth position in the sequence.
Recursive rule: Following the first term, each term is 4 more than the previous term.

F&A – 7 Expresses generalization or rule using words or symbols:

Expressing a rule for a term or the n^{th} term of a pattern means to write a rule using words, expressions, or equations that can be applied to find the value for an identified term or for any term.

Note: Students are asked to express the rule using words and/or symbols (expressions or equations) depending upon the grade level and the type of pattern (linear or nonlinear). See Pattern Summary Table ***F&A – 9*** (page 17).

Example 7.1: The table below shows the relationship between the number of tickets sold for a movie at $7 per ticket and the total revenue generated by ticket sales.

Next case

100th term

n^{th} case

Number of Tickets Sold	Total Revenue ($)
1	7
2	14
3	21
4	28
5	
6	
7	
⋮	⋮
10	
⋮	⋮
100	
⋮	⋮
n	

Generalizes in words: A non-recursive rule for the pattern in the table above is multiply the number of tickets sold by the cost of each ticket ($7) to generate the total revenue.

Using symbols in an algebraic expression: A non-recursive algebraic expression is $7s$, where s is the number of tickets sold.

Generalizes using symbols in an equation: A non-recursive explicit equation is $r = 7s$, where r is the total revenue for s tickets sold.

F&A – 8 **Generalizes a pattern to find a specific case:** To find the value of the 1000^{th} element in the sequence 4, 7, 10, 13,… in the table below, it is efficient to first find a non-recursive rule that works for the elements in the set that are given and then apply the rule to the 1000^{th} case.

Example 8.1: If the input is 1000, what is the corresponding output?

Input	Output
1	4
2	7
3	10
4	13
⋮	⋮
1000	?

Recursive rule: Add 3 to the previous output. **Note**: It is inefficient to find the 1000^{th} term (output) of the pattern by iteratively adding 3 until you get to the 1000^{th} case.
Non-recursive explicit equation: Output = (3 • Input) + 1. This rule can be applied to any case of the pattern. The Output for the 1000^{th} Input is 3 • 1000 + 1 = 3001.

F&A – 9 **Pattern Summary Table by Grade Level:** Type of pattern, action, and representations of patterns required at different grades.

- An "or^{sc}" means that the requirement is student choice (e.g., Write rule to find the next case using words or^{sc} symbols.).

Grade	Type of Pattern	Patterns represented in	Action	Representation of Pattern
K	Non-numeric	Models, sequences, movement	Identify and extend to specific cases; Translate repeating patterns across formats	
1	Non-numeric Numeric	Models, tables, sequences	Identify and extend the pattern to the next one, two, or three elements; Translate repeating patterns across formats; Find a missing element of a pattern	
2	Linear Non-numeric	Models, tables, sequences	Extend pattern to next element; Find a missing element of a pattern	
3	Linear Non-numeric	Models, tables, sequences	Extend pattern to next one, two or three elements; Find a missing element of a pattern	
4	Linear Nonlinear	Models, tables, sequences	Extend pattern to specific cases in pattern (including and beyond the next three cases)	Writes a rule to find the next case using words or^{sc} symbols (recursive or^{sc} non-recursive rule)
5	Linear Nonlinear	Models, tables, sequences, problem situations	Extend pattern to specific cases in pattern (including and beyond the next three cases)	Writes a rule to find specific cases (beyond the next case) of linear patterns using words or^{sc} symbols. (recursive or^{sc} non-recursive rule)
6	Linear Nonlinear	Models, tables, sequences, graphs, problem situations	Extend pattern to specific cases in pattern (including and beyond the next three cases); Writes an expression to express the generalization of a linear pattern	Writes a rule to find specific cases of a nonlinear pattern; Write an expression or^{sc} equation to express a generalization of a linear pattern. (recursive or^{sc} non-recursive rule)
7	Linear Nonlinear	Models, tables, sequences, graphs, problem situations	Extend pattern to specific cases in pattern (including and beyond the next three cases); Extend to cases that require a generalization (e.g., What is the 100th case?); Generalize linear and writes an expression to express the generalization of a nonlinear pattern	Write an equation to generalize a linear pattern (non-recursive explicit equation); Writes an expression or^{sc} equation to express a generalization of a nonlinear pattern (recursive or^{sc} non-recursive rule)
8	Linear Nonlinear	Models, tables, sequences, graphs, problem situations	Extend pattern to specific cases in pattern (including and beyond the next three cases); Extend to cases that require a generalization (e.g., What is the 100th case?) Generalizes linear and nonlinear patterns	Write an expression or equation to express generalization of a linear pattern (non-recursive explicit equations); Write an equation using words or^{sc} symbols to generalize a nonlinear pattern (non-recursive explicit equations).

Resource Material Prototype

Section 2: Linear and Nonlinear Relationships and Functions

NECAP: M(F&A) –X – 2
Vermont: MX: 20

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Concrete situations	31	*F&A – 20*
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Describes the meaning of slope and intercept in concrete situations	26	*F&A – 15*
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Domain of a function	41	*F&A – 23*
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Informally determines slope	27	*F&A – 16*
Intercept	28	*F&A – 17*
Linear relationships	19	*F&A – 10*
Nonlinear relationships	30	*F&A – 18*
Non-proportional linear relationships ($y = mx + b$)	21	*F&A – 12*
Proportional linear relationships ($y = kx$)	19	*F&A – 11*
Range of a function	43	*F&A – 24*
Slope	24	*F&A – 14*
Solves problems involving linear relationships	23	*F&A – 13*

Note: Arrowheads on graphs are used throughout this section to indicate that the data go beyond the values shown. However, it does not necessarily mean that the data extend indefinitely.

F&A – 10 **Linear relationships:** Linear relationships include proportional relationships (grades 3 – High School) and non-proportional linear relationships (grades 6 – High School). Linear relationships can be represented in equations, tables, graphs, or expressed as rules using words.

F&A – 11 **Proportional linear relationships:** Proportional linear relationships can be expressed in the form $y = kx$ where k represents the *constant rate of change* or slope of the line. The graph of a proportional relationship is a straight line that passes through the origin.

Example 11.1 – Proportional linear relationships ($y = kx$):

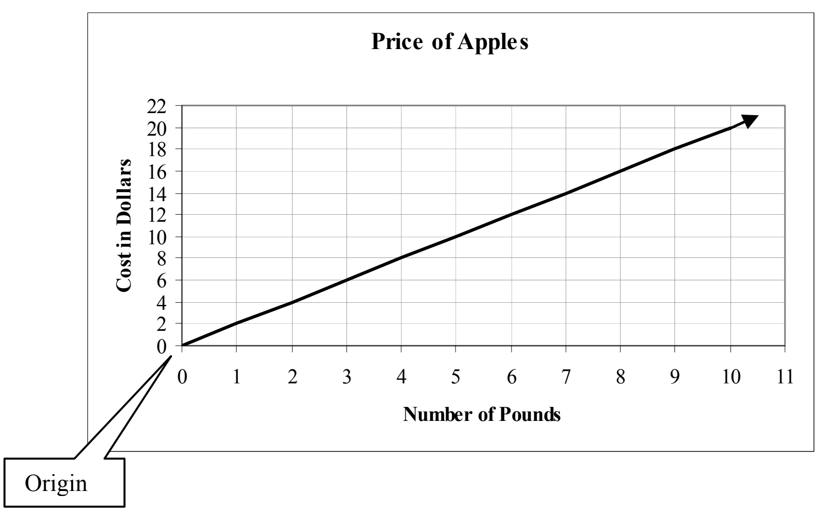
A farmer sells apples for $2.00 per pound.

Rule in words: If apples cost $2.00 per pound, then the total cost of purchasing apples is $2.00 multiplied by the number of pounds purchased.

Rule in equation: If the price of apples per pound (k) is $2.00, the total cost ($y$) of purchasing apples at $2.00 per pound is calculated by multiplying the number of pounds of apples (x) purchased by $2.00. Therefore, the equation in $y = kx$ form that represents the relationship between the total cost of the apples and the number of pounds of apples is $y = 2x$.

Apples for Sale
$2.00 per pound

Graph



Table

Price of Apples	
Number of Pounds of Apples (x)	**Cost (y)**
0	0
1	$2.00
2	$4.00
3	$6.00
4	$8.00
5	$10.00

In this example, to find the cost, the number of pounds is always multiplied by $2.00 which is the constant rate of change (k).

(Definition *F&A – 11* continued on following page)

Example 11.2 – Discrete proportional linear relationships ($y = kx$):

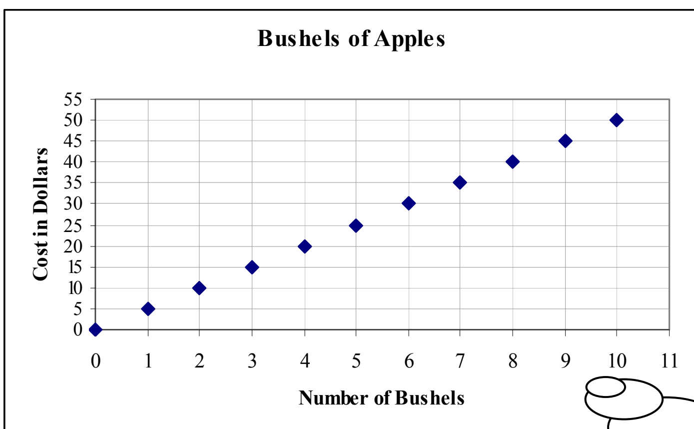
A farmer sells apples for $5.00 per bushel. (Bushels are not sold in fractional parts.)

Rule in words: If apples cost $5.00 per bushel, then the total cost of purchasing apples is $5.00 multiplied by the number of bushels purchased.

Rule in Equation: If the price of apples per bushel (k) is $5.00, the total cost ($y$) of purchasing apples at $5.00 per bushel is calculated by multiplying the number of bushels of apples purchased (x) by $5.00. Therefore, the equation in $y = kx$ form that represents the relationship between the total cost of the apples and the number of bushels of apples is $y = 5x$.

Apples for Sale
$5.00 per bushel

Graph



Table

Price of Apples	
Number of Bushels (x)	**Cost (y)**
0	0
1	$5.00
2	$10.00
3	$15.00
4	$20.00
5	$25.00

Discrete graph: Since there is no opportunity for the purchaser to buy a fractional part of a bushel, the graph of this linear relationship is discrete as opposed to continuous. Connecting two points on the graph would indicate that fractional parts of bushels are possible, which is not allowed in this case.

F&A – 12 **Non-proportional linear relationships:** Non-proportional linear relationships can be expressed in the form $y = mx + b$, where $b \neq 0$, m represents the *constant rate of change* or slope of the line, and b represents the y-intercept. The graph of a non-proportional linear relationship is a straight line that does **not** pass through the origin.

Example 12.1 – Non-proportional linear relationships ($y = mx + b, b \neq 0$):

Ajax Taxicab Company charges a flat fee of \$1.00 plus \$0.30 per mile to ride in a cab. (Assumption: The flat fee is incurred as soon as you enter the cab.)

Rule in words: To determine the cost of an Ajax Taxicab ride, multiply the number of miles traveled by \$0.30, and then add \$1.00 (the flat fee) to the product.

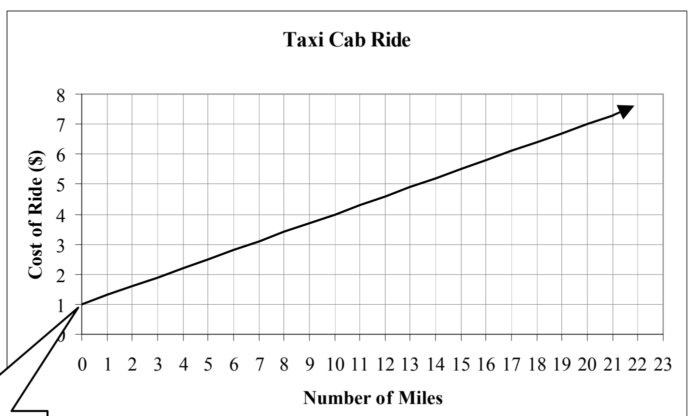
Rule in Equation: If y represents the total cost of an Ajax Taxicab ride of x miles, then the relationship can be expressed as an equation in the form of $y = mx + b$, where m represents the cost per mile (\$0.30/mile) and b represents the flat fee (\$1.00).

Total Cost = Cost per Mile • Number of Miles + Flat Fee

$$y = 0.30 \cdot x + 1.00$$

OR… $y = 0.30x + 1.00$

Graph



y-intercept – represents flat fee of \$1.00 for traveling 0 miles (for getting into the cab).

Table

Price of Taxi Cab Ride	
Number of Miles (x)	**Cost (y)**
0	\$1.00
1	\$1.30
2	\$1.60
3	\$1.90
4	\$2.20
5	\$2.50

In this example, to find the cost of a cab ride the number of miles traveled is multiplied by \$0.30, and then the flat fee (\$1.00) is added to the product.

(Definition *F&A – 12* continued on following page)

Example 12.2 – Non-proportional linear relationships (with negative slope):

A 10-inch candle burns at a *constant* rate of 1 inch per hour.

Rule in words: To determine the height of the candle multiply the number of hours that the candle burns by 1 inch per hour, and subtract the product from the candle's initial height (10 inches).

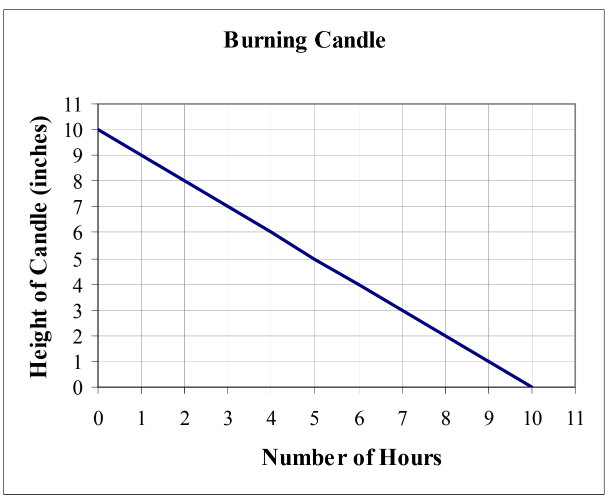
Rule in Equation: If y represents the height of the candle after x hours of burning, then the relationship can be expressed as an equation in the form $y = mx + b$, where m represents the rate at which the candle burns (1 inch per hour) and b represents the initial height of the candle (10 inches).

Height of Candle = Rate at which it Burns · Number of Hours Burned + Initial Height

$$y \quad = \quad -1 \quad \cdot \quad x \quad + \quad 10$$

OR… $y = -1x + 10$ or $y = 10 - 1x$

Graph

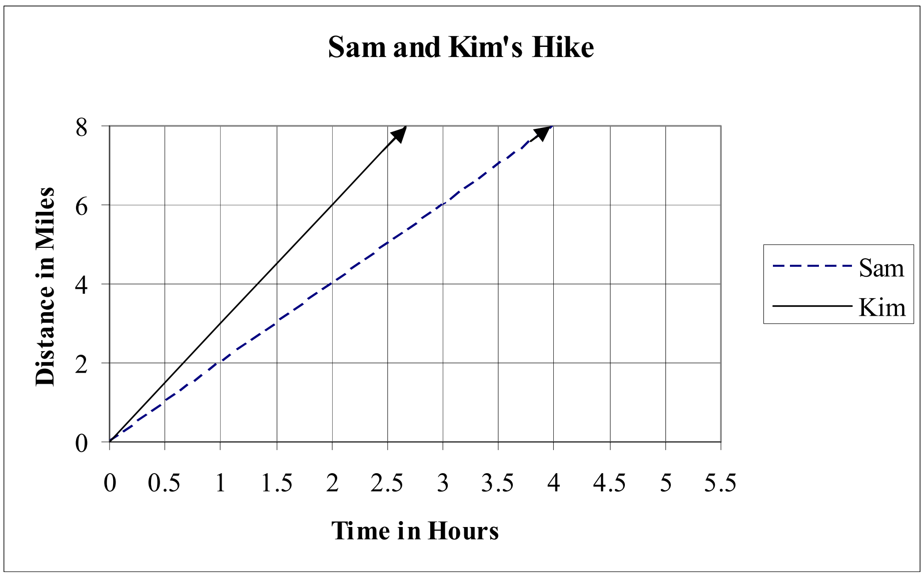


Table

Burning Candle	
Number of hours candle burns (x)	**Height of candle in inches (y)**
0	10
1	9
2	8
3	7
4	6
5	5
6	4
7	3
8	2
9	1
10	0

F&A – 13 **Solves problems involving linear relationships:** Students will solve problems in which they have to interpret graphs or tables of values involving linear relationships.

Example 13.1: Kim is going on a 12-mile hike. Sam is going on a 10-mile hike. If Kim and Sam start hiking at the same time and continue to hike at the rates shown in the graph below, who will finish their hike first, and by how much time? Show your work.



Using the graph we can see that . . .

- Kim has traveled 6 miles in 2 hours. Therefore, she is hiking at a rate of 3 mph.
- Sam has traveled 4 miles in 2 hours. Therefore, he is hiking at a rate of 2 mph.

Answer: Kim is traveling at 3 mph so she will finish the 12-mile hike in 4 hours. Sam is traveling at 2 mph, so he will finish the 10-mile hike in 5 hours. Kim will finish first by 1 hour.

F&A –14 **Slope:** Slope is a measure of steepness and represents a rate of change. Since the steepness of a line does not vary, the slope of any line is a constant (with the exception of a vertical line which has an undefined slope). Lines that rise from left to right have a positive slope. Lines that fall from left to right have a negative slope. Horizontal lines have a slope of 0.

The slope of a line is determined by calculating the ratio of the vertical change to horizontal change between *any* two points on the line.

Typical ways of representing slope are:

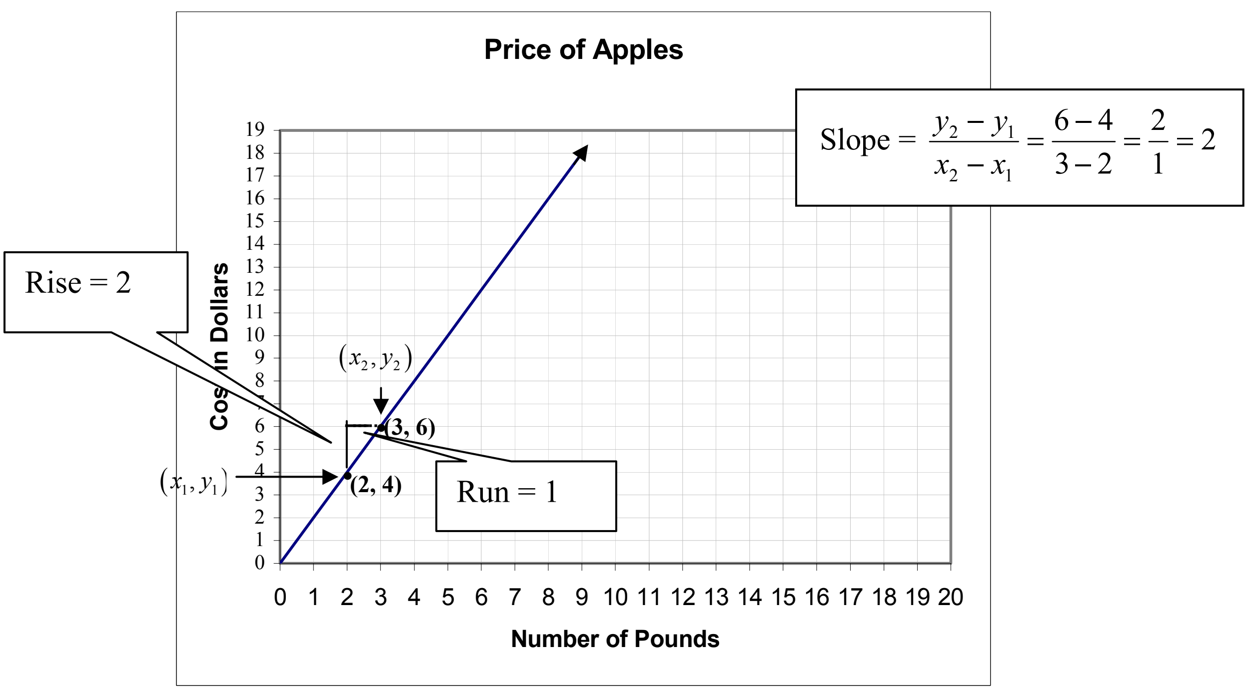
$$\text{Slope} = \frac{\text{Vertical Change}}{\text{Horizontal Change}}$$

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}}$$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } (x_1, y_1) \text{ and } (x_2, y_2) \text{ are two points on the line}$$

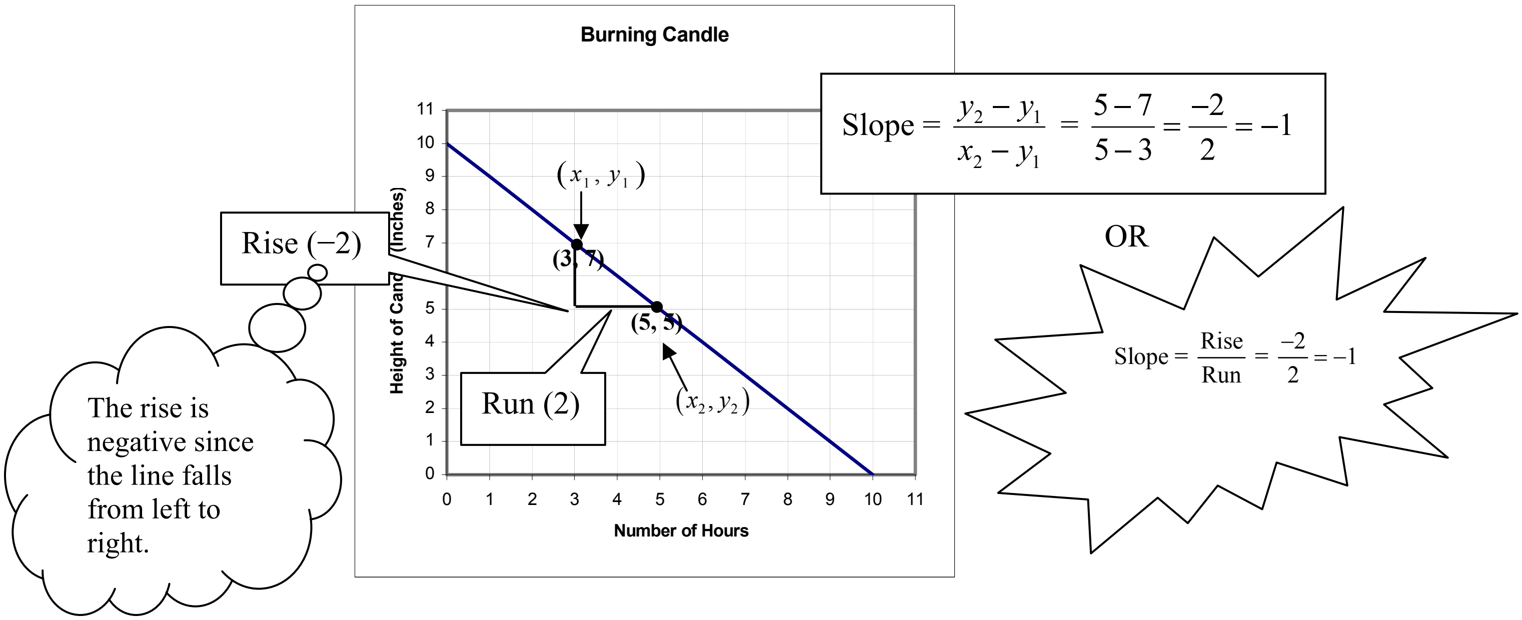
Example 14.1 – Positive Slope: The total cost of buying apples increases at a constant rate as the total weight of apples purchased increases.

For example, given the points on the graph below, the rise is 2 because the cost of the apples purchased has increased by $2.00 (from $4.00 to $6.00). Similarly, the run is 1 because the number of pounds of purchased apples has increased by 1 (from 2 lb to 3 lb) between the identified points. Therefore, the slope is 2, or $2.00 per pound.



(Definition *F&A – 14* continued on following page)

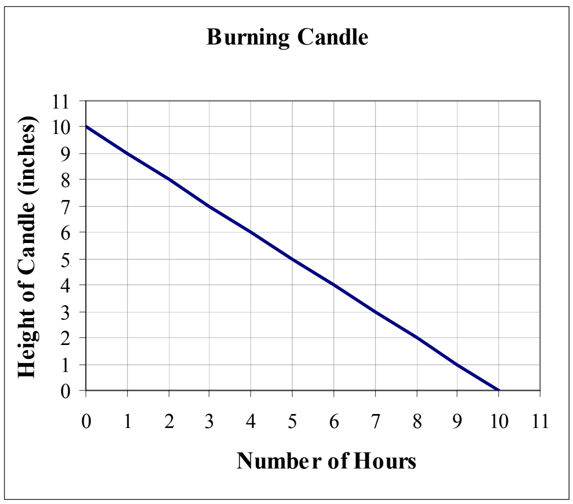
Example 14.2 – Negative Slope: For every hour that a candle burns it decreases in height by 1 inch. The slope is – 1.

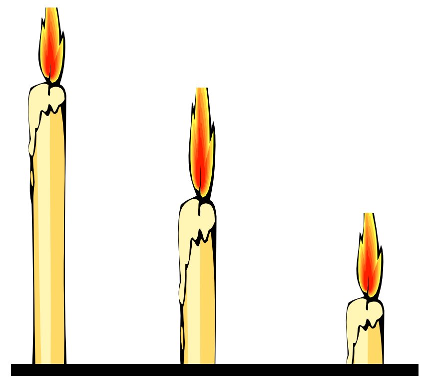


Note: Even though Examples 14.1 and 14.2 show formal calculations for determining slope, the expectations in the GLEs at middle school focus on **demonstrating an informal understanding of slope** – "…describe the meaning of slope in concrete situations, and informally determine slope…" as illustrated in ***F&A – 15*** and ***F&A – 16***.

F&A – 15 Describes the meaning of slope and intercept in concrete situations:
Students will interpret the meaning of the slope of a line and the *x*- and *y*-intercepts relative to a problem situation.

Example 15.1: What is the meaning of the slope in this situation? What is the meaning of the *y*-intercept in this situation?

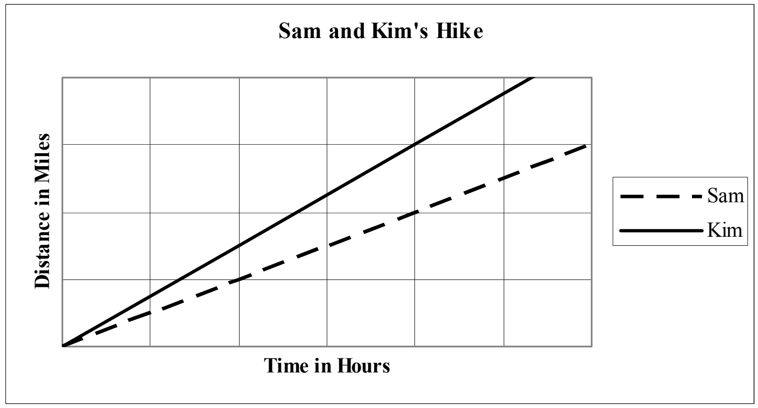




Answer: The slope represents that the candle burns at a constant rate of 1 inch per hour and the *y*-intercept represents the initial height of the candle (10 inches).

(Definition *F&A – 15* continued on following page)

Example 15.2: Who is hiking at a faster speed, Kim or Sam? Explain how you know.

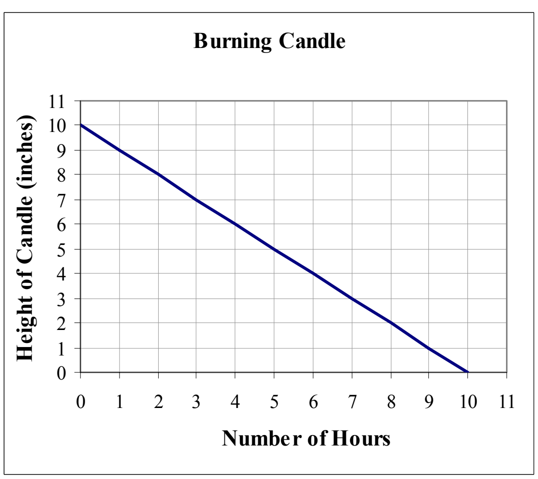


Answer: Slope is a measure of steepness and represents a rate; the steeper the slope, the greater the rate of change. Therefore, Kim is hiking faster because the slope of the line representing Kim's hike is steeper than the slope of the line representing Sam's hike.

Note: The rate at which each person is hiking is represented by the slope of the corresponding line. Slope represents a ratio of vertical change to horizontal change. In this situation the ratio of vertical change to horizontal change ($\frac{\text{Change in distance}}{\text{Change in time}}$) results in a rate in miles/hour.

F&A – 16 **Informally determines slope:** Informally determines slope means that students are **not** required to apply the slope formula to determine the slope (e.g., determine the slope from a graph or table of values as in Example 16.1 compared to a more formal calculation as shown in Example 14.2).

Example 16.1: What is the rate at which the candle is burning?

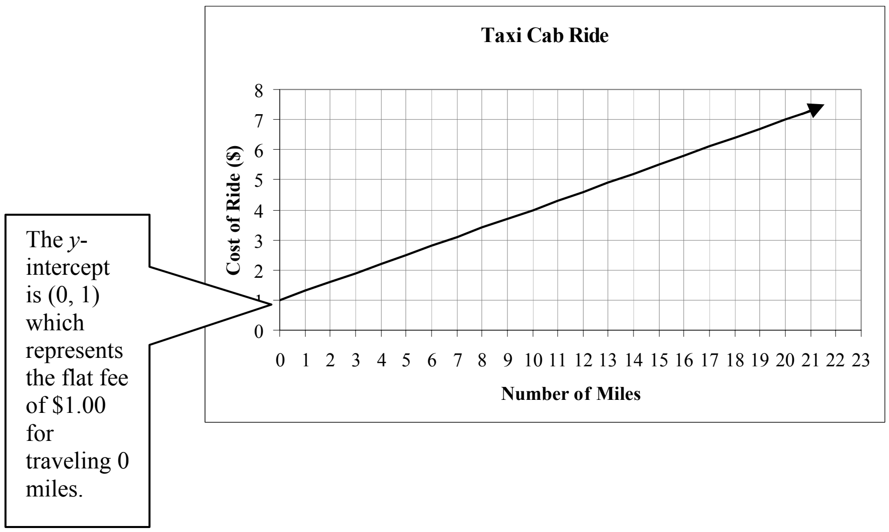


Burning Candle	
Number of hours candle burns (x)	**Height of candle in inches (y)**
0	10
1	9
2	8
3	7
4	6
5	5
6	4
7	3
8	2
9	1
10	0

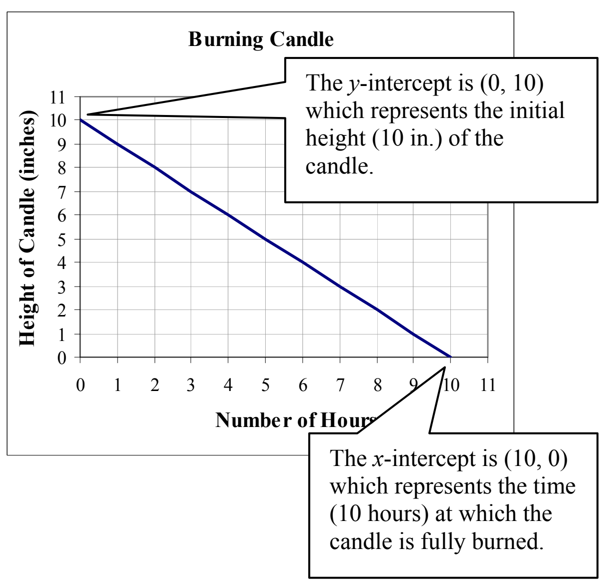
Answer: For every hour that the candle burns it decreases in height by 1 inch. The slope is – 1, or the candle burns at a rate of 1inch per hour.

F&A – 17 **Intercept**: An intercept is a point where a line intersects either of the axes on a coordinate graph.

Example 17.1 – The Cab Ride: (See Example 12.1 for problem context.)



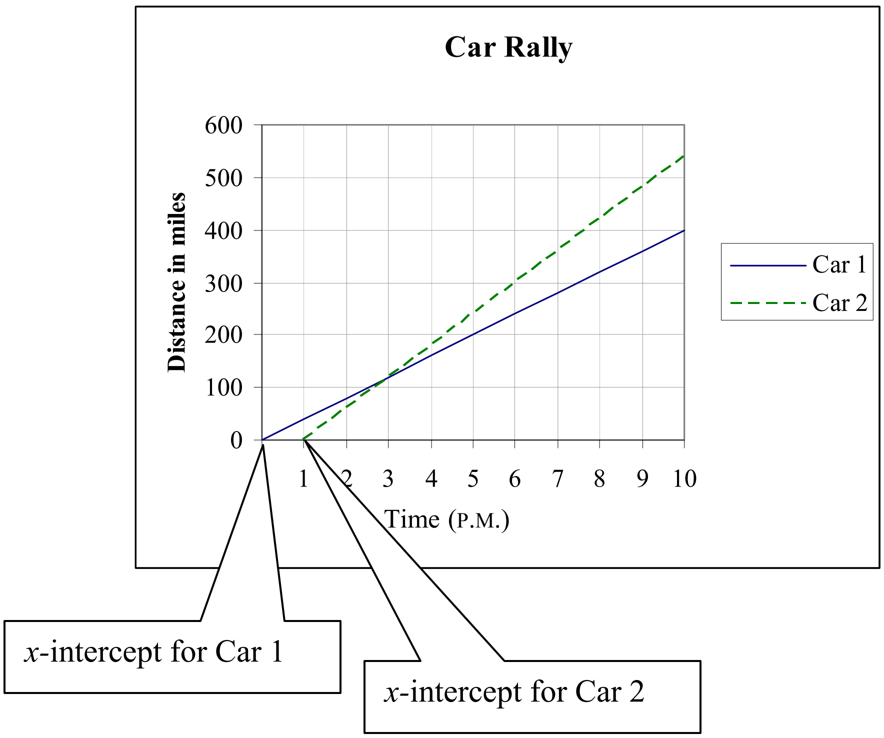
Example 17.2 – The Burning Candle: (See Example 12.2 for problem context.)



(Definition *F&A – 17* continued on following page)

Example 17.3 – A Car Rally:

The x-intercept is the location where a line intersects the x-axis. In this example, Car 1 begins the rally at 12:00 p.m. Its x-intercept is (12:00 P.M., 0). Car 2 begins the rally at 1:00 P.M. Its x-intercept is (1:00 P.M., 0).

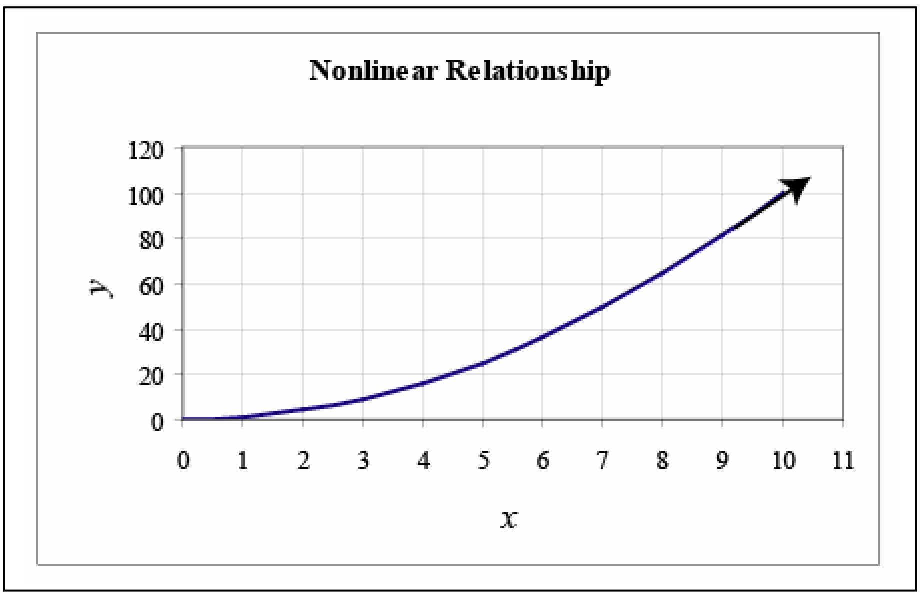


Car Rally		
	Distance (miles)	
Hour	**Car 1**	**Car 2**
12 P.M.	0	
1 P.M.	40	0
2 P.M.	80	60
3 P.M.	120	120
4 P.M.	160	180
5 P.M.	200	240
6 P.M.	240	300
7 P.M.	280	360
8 P.M.	320	420
9 P.M.	360	480
10 P.M.	400	540

F&A – 18 **Nonlinear relationships:** Nonlinear relationships are characterized by variable rates of change. The graphs of nonlinear relationships are **NOT** straight lines.

Nonlinear relationships are not proportional relationships.

Example 18.1: $y = x^2$



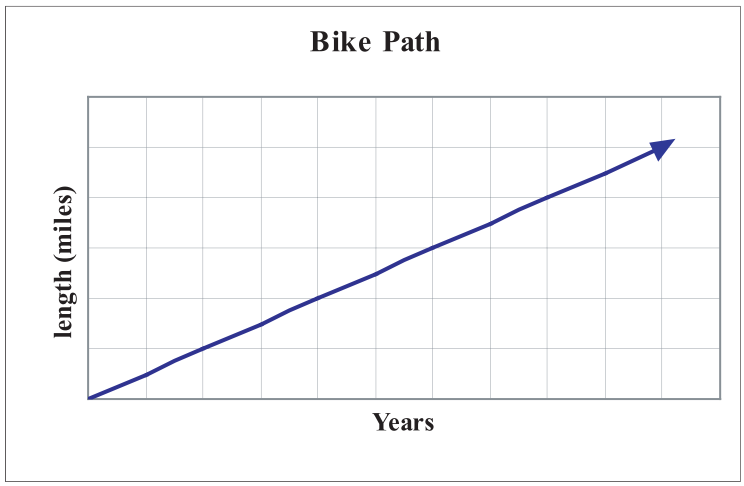
x	*y*
0	0
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

Note: The graph shown here is for $x \geq 0$ and the table only shows whole number inputs.

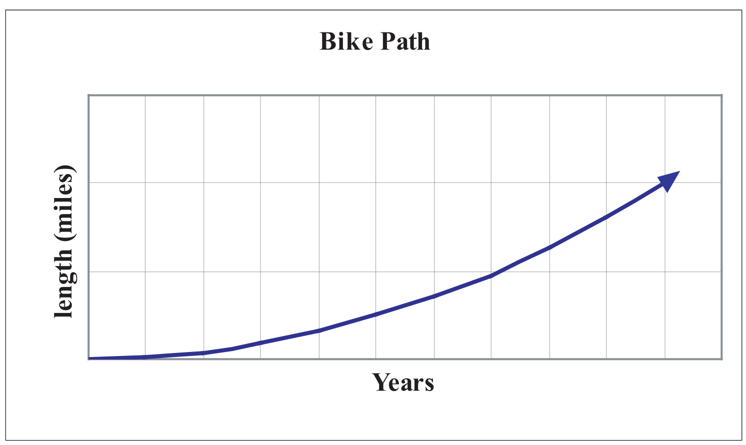
F&A – 19 **Distinguishes between constant and varying rates:** Analyze a table of values or graph to determine if the relationship represented is either linear (constant rate of change) or nonlinear (varying rate of change). Provide a justification for the choice.

Example 19.1: Which graph best represents the length of a bike path if the length of the path is increased 10 miles each year? Provide an explanation for your choice.

A.



B.



Answer: Graph "A" best represents the length of the bike path. The length of the bike path is increased at a constant rate each year and results in a line, not a curve.

F&A – 20 **Concrete situations:** The expression "concrete situations" means that problems should be cast in contexts most students at a grade level have had experience with. *Burning Candle*, *Taxi Cab Ride*, *Price of Apples*, *Sam and Kim's Hike*, and *CarRally* are examples of concrete situations (see examples 11.1–17.3).

Going beyond the K – 8 GLEs: Definitions *F&A – 21-24* (function, dependent and independent variables, domain of a function, and range of a function are included in this section to answer the question – "Where does the Functions and Algebra strand go next?" **However,** the concepts addressed in this section will <u>not</u> be assessed on the K–8 NECAP.

F&A – 21 **Function:** A function is a relation (a set of ordered pairs) where each first coordinate has exactly one (*unique*) corresponding second coordinate.

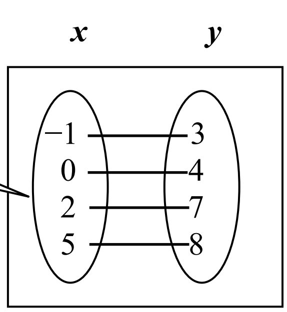
Examples 11.1–19.1 given in this section prior to this statement are all relations that are also functions. Functions can be represented as sets of ordered pairs, in tables or graphs, or expressed as equations and can be linear or nonlinear.

Typically, the first coordinate in an ordered pair is called the x-coordinate. The second coordinate is called the y-coordinate. In a function we say, "each x has exactly one corresponding y."

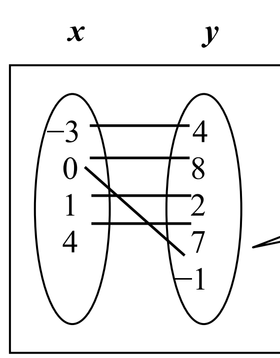
Examples 21.1 – 21.5 include ordered sets that are functions and ordered sets that are NOT functions.

Example 21.1 – Function:
Consider the set of ordered pairs S, where $S = \{(-1, 3), (0, 4), (2, 7), (5, 8)\}$.

S is a **function** since each first coordinate has exactly one second coordinate.



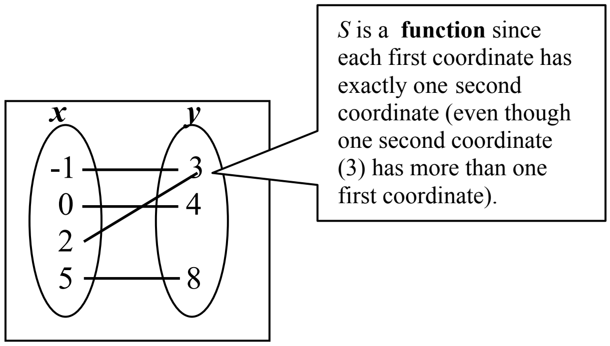
Example 21.2 – <u>Not</u> a function:
Consider the set of ordered pairs S, where $S = \{(-3, 4), (0, 8), (1, 2), (4, 7), (0, -1)\}$.



S is **<u>NOT</u> a function** since one first coordinate (0) corresponds to different second coordinates.

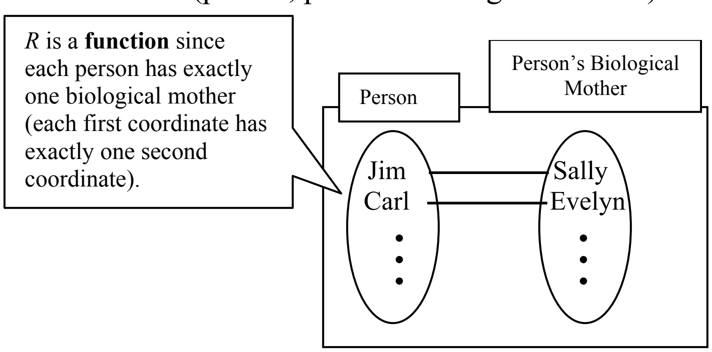
(Definition *F&A –21* continued on following page)

Example 21.3 – Function: Consider the set of ordered pairs S, where $S = \{(-1, 3), (0, 4), (2, 3), (5, 8)\}$.

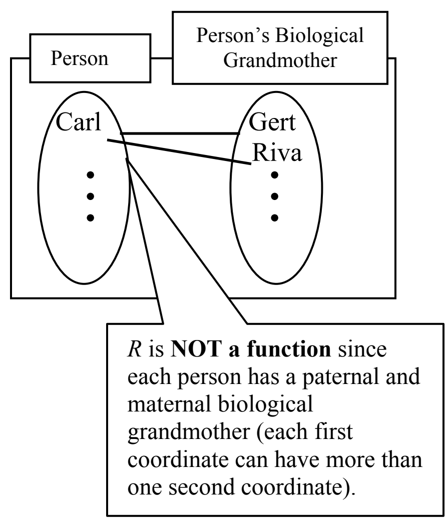


Note: In functions, it is possible for one second coordinate to have more than one first coordinate.

Example 21.4 – Function:
Consider the set R consisting of all the ordered pairs of the form (person, person's biological mother).



Example 21.5 – Not a Function:
Consider the set R consisting of all the ordered pairs of the form (person, person's biological grandmother).



(Definition *F&A –21* continued on following page)

Example 21.6 – Functions as equations: Consider the function *f* consisting of all ordered pairs where the second coordinate is twice the first coordinate, where the first coordinate is restricted to be a natural number.

x	y
1	2
2	4
3	6
4	8
5	10
6	12
$\vdots$	$\vdots$

In this case, *f* consists of an infinite number of ordered pairs; hence, it is impossible to write down all members of *f*. In cases like this, it is typical to express the function as an equation that explicitly indicates the relationship between the second coordinate and the first coordinate.

If y represents the second coordinate and x represents the first coordinate, then *f* can be expressed as $y = 2x$ (recall x can only be a natural number).

Example 21.7 – Functional notation:

Consider Example 21.6. Rather than writing statements such as:

when $x = 1$, $y = 2$,
when $x = 3$, $y = 6$, and
when $x = 6$, $y = 12$,

it is convenient to have a short cut notation that indicates the corresponding x and y values for the function *f*. This notation is illustrated below.

In Words	In Symbols	Additional ways to interpret relationships
When $x = 1$, $y = 2$	$f(1) = 2$	*f* maps 1 to 2
When $x = 3$, $y = 6$	$f(3) = 6$	*f* maps 3 to 6
When $x = 6$, $y = 12$	$f(6) = 12$	*f* maps 6 to 12

The examples in this table shows ways to indicate that (1, 2), (3, 6), and (6, 12) are all elements of function *f* (which consists of all ordered pairs where the second coordinate is twice the first coordinate).

We may also write $f(x) = 2x$ in place of writing $y = 2x$.

(Definition *F&A – 21* continued on following page)

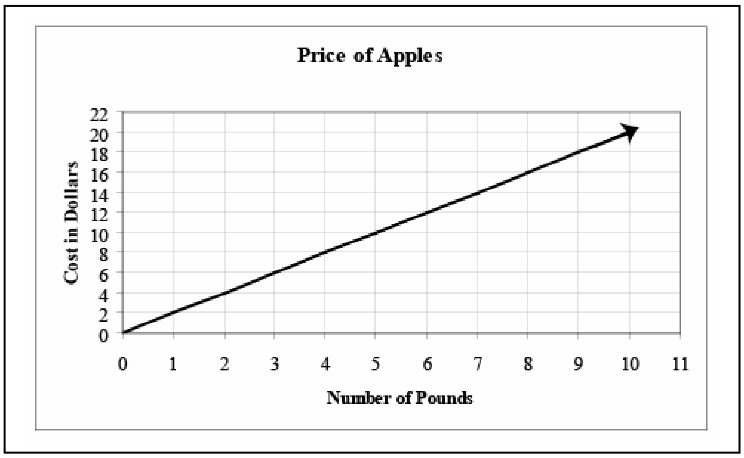
Example 21.8, 21.9 and 21.10 – Other examples of functions:

Example 21.8: (See Example 11.1 for context.) Since each possible weight of purchased apples (in pounds) has only one possible price, the relationship between the total cost of the apples purchased and the number of pounds of apples purchased is a functional relationship.

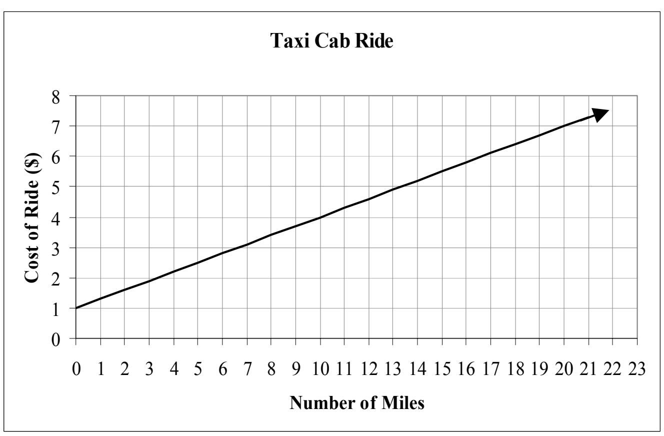
$$f(x) = kx$$

$$f(\text{number of pounds of apples}) = \text{cost per pound} \cdot \text{the number of pounds purchased}$$

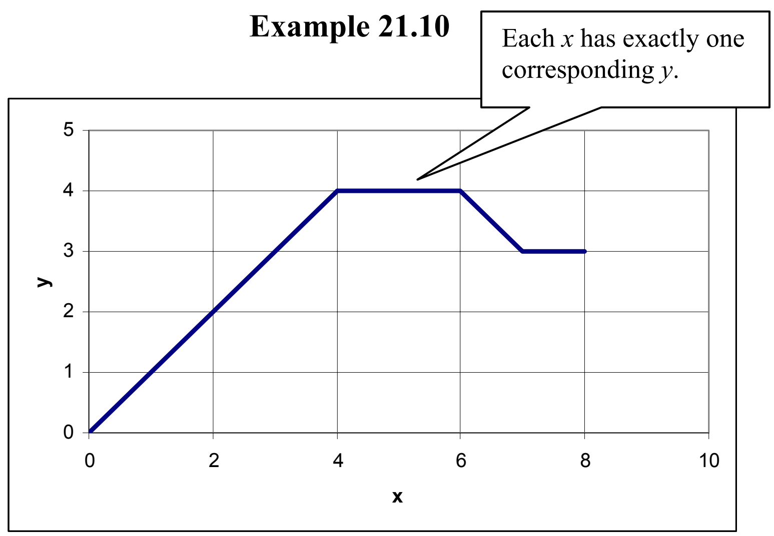
$$f(x) = 2x$$



Example 21.9:
(See Example 12.1 for context.)



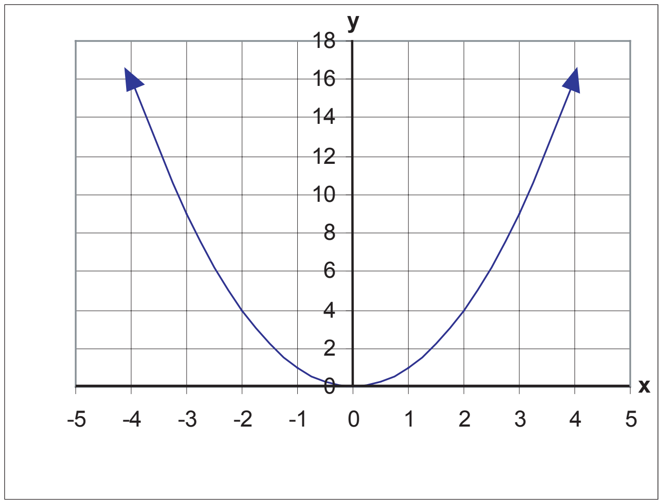
Example 21.10



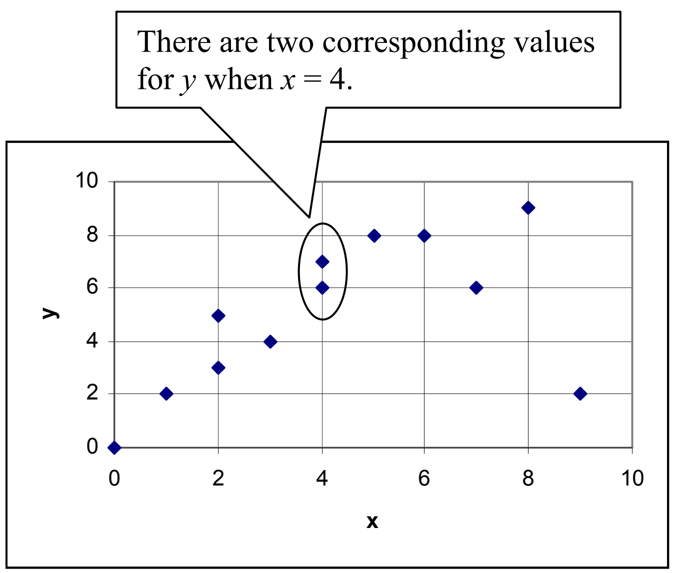
(Definition *F&A – 21* continued on following page)

Example 21.11 – Nonlinear function:

$f(x) = x^2$



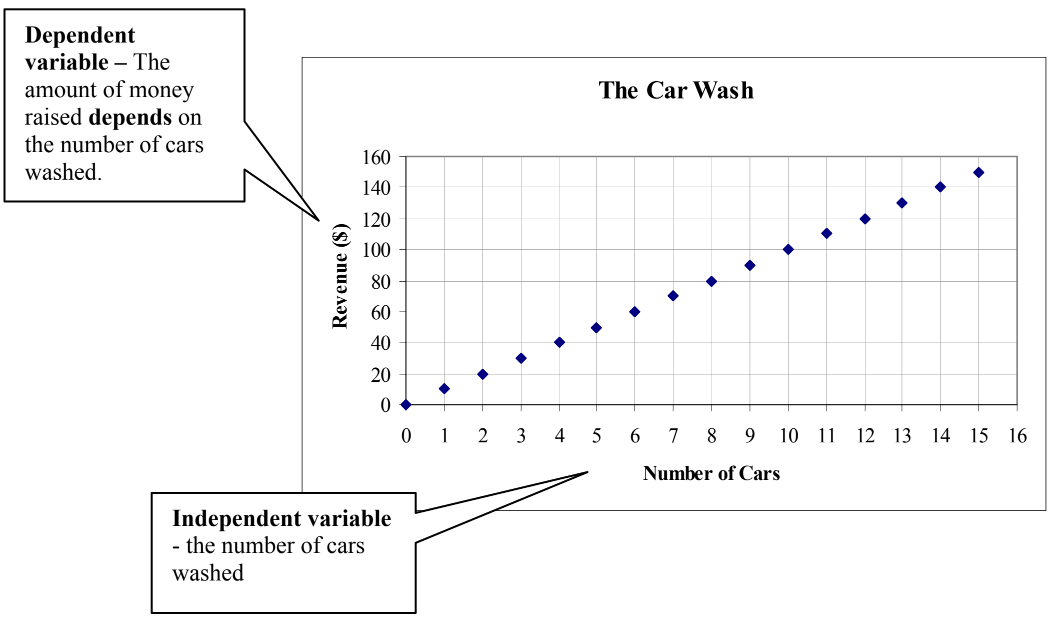
Example 21.12: This graph is **NOT** a function because there is **not** a single value for every value of *x* on the graph.



F&A – 22 **Dependent and independent variables:** The terms dependent and independent variables describe how the variables in a relationship are associated to each other. The value of the dependent variable, as its name implies, is determined by the value of the independent variable.

Example 22.1: A soccer team is holding a car wash to raise money. They are charging a flat rate of $10.00 for each car that gets washed.

In this situation, the amount of money raised (revenue) is **dependent** on the number of cars washed. Therefore, the amount of money raised (revenue) is the dependent variable, and the number of cars washed is the independent variable.



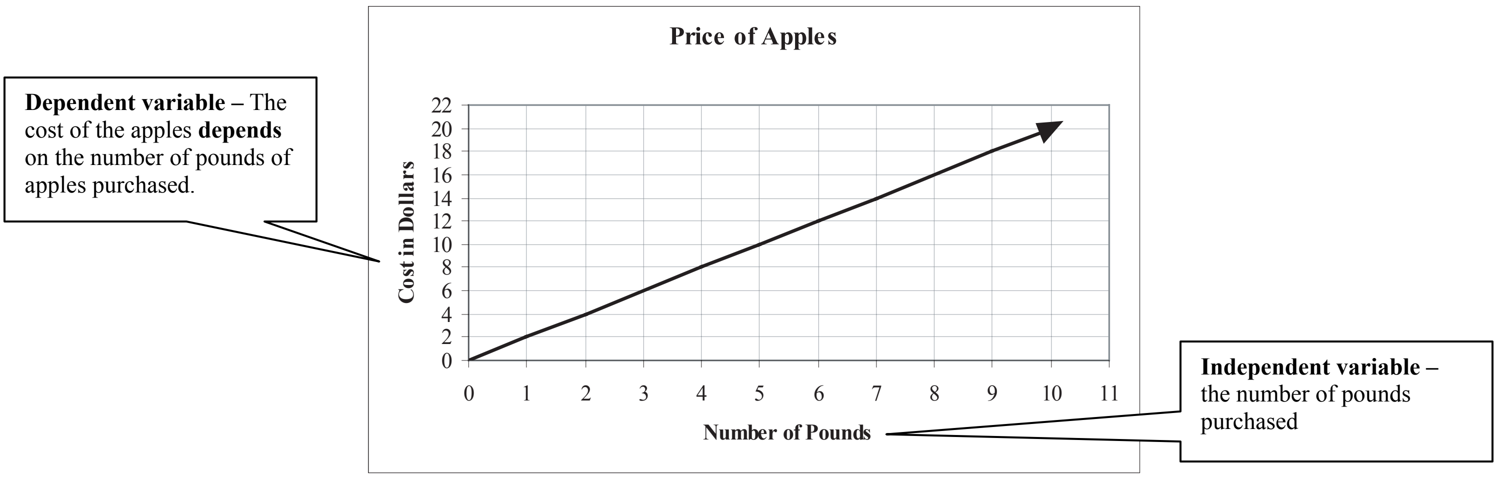
Note: In a situation involving one independent variable and one dependent variable, typically the independent variable is associated with the values on the horizontal axis and the dependent variable is associated with the variables on the vertical axis.

Sometimes distinguishing the dependent variable from the independent variable is not as clear cut as in this example; Examples 22.2 and 22.3 help illustrate this dilemma.

(Definition *F&A – 22* continued on following page)

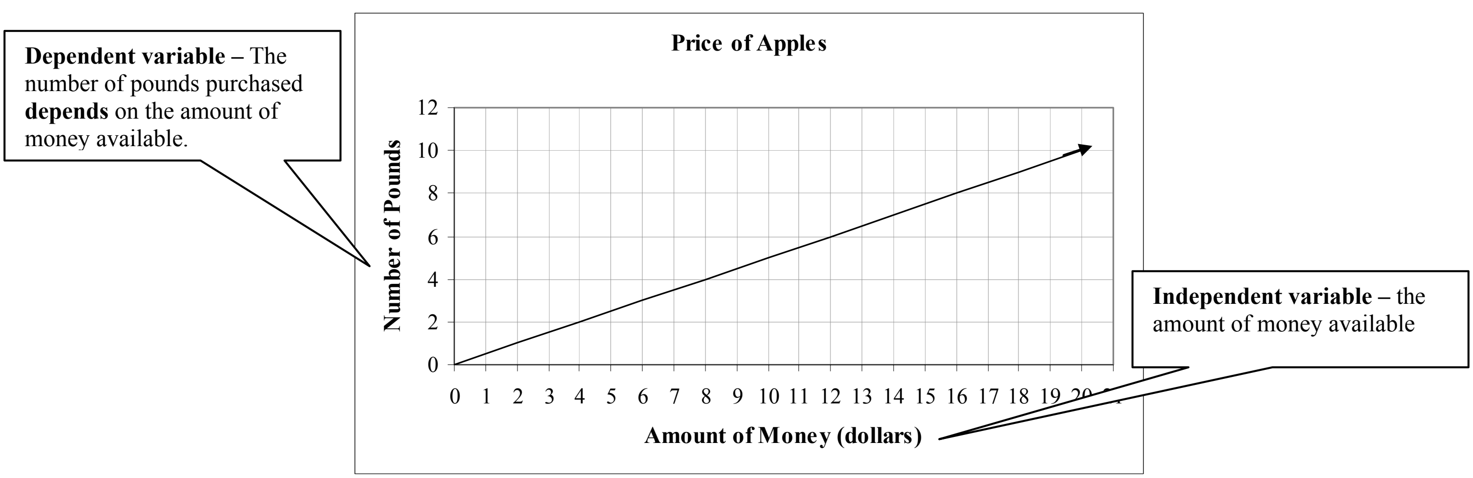
Example 22.2 – Unlimited resources ($): Apples are sold for $2.00 a pound.

If people had unlimited resources when purchasing apples, then the total cost of apples purchased would be **dependent** upon the number of pounds of apples purchased.



Example 22.3 – Limited Resources ($): Apples are sold for $2.00 a pound.

However, most people do not go shopping with an unlimited amount of money. In this case, the amount (number of pounds) of apples purchased **depends** upon how much money a shopper has to spend.



(Definition *F&A – 22* continued on following page)

Example 22.4 – In equations:

Note: Examples 22.1 – 22.3 are examples that can be expressed as equations.

As shown in Example 22.1, a soccer team is holding a car wash to raise money. They are charging a flat rate of $10.00 for each car that gets washed.

An equation that represents this situation is…

Independent variable – the number of cars washed

Dependent variable – The amount of money raised **depends** on the number of cars washed.

$$y = 10x$$

***y* represents** the total revenue earned.

***x* represents** the number of cars.

(Definition *F&A – 22* continued on following page)

Example 22.5: Example 22.2 and 22.3 are interesting cases. Both situations appear the same – “Apples are sold for $2.00 a pound.” However, in Example 22.2 the resources ($) are unlimited, and in Example 22.3 the resources ($) to buy the apples are limited. These differences result in different equations to represent each of the situations.

Apples are sold for $2.00 a pound

Unlimited resources (Example 22.2)

y represents the total cost (dollars) of apples purchased.
x represents the number of pounds of apples purchased.

Dependent variable – The total cost of apples purchased is **dependent** on the number of pounds of apples purchased.

$$y = 2x$$

Independent variable – the number of pounds of apples purchased

Limited resources (Example 22.3)

y represents the total number of pounds of apples purchased
x represents the money available to purchase apples

Dependent variable – The total number of pounds that can be purchased is **dependent** on the amount of money available to purchase the apples.

$$y = \frac{1}{2}x$$

Independent variable – the amount of money available to purchase the apples

Note: In a situation with one independent and one dependent variable, it is standard to label the independent variable as “*x*” and the dependent variable as “*y*.” However, students should develop flexibility with different labeling schemes, as variables are often given different names. It is the **relationship** between the variables, **not the labels** of the variables that is important.

In Examples 22.2 and 22.3 some assumptions are made about the availability of resources (both money and pounds of apples available). In order to model real life situations, it is important to consider any restrictions on the variables involved. This requires an understanding of domain and range. (See ***F&A – 23***, ***F&A – 24***.)

F&A – 23 **Domain of a function:** The domain of a function *f* is the set consisting of all of the first coordinates of all the ordered pairs that are members of *f*.

Example 23.1: $S = \{(-1, 3), (0, 4), (2, 7), (5, 8)\}$

The domain of S is $\{-1, 0, 2, 5\}$.

Example 23.2: Consider the function *f* consisting of all ordered pairs where the second coordinate is twice the first coordinate and the first coordinate is restricted to be a natural number.

x is restricted to a natural number.

x	*y*
1	2
2	4
3	6
4	8
5	10
6	12
⋮	⋮

The domain, in this case, is the set of natural numbers or shown as $\{1, 2, 3, 4, 5, \ldots\}$.

(Definition *F&A – 23* continued on following page)

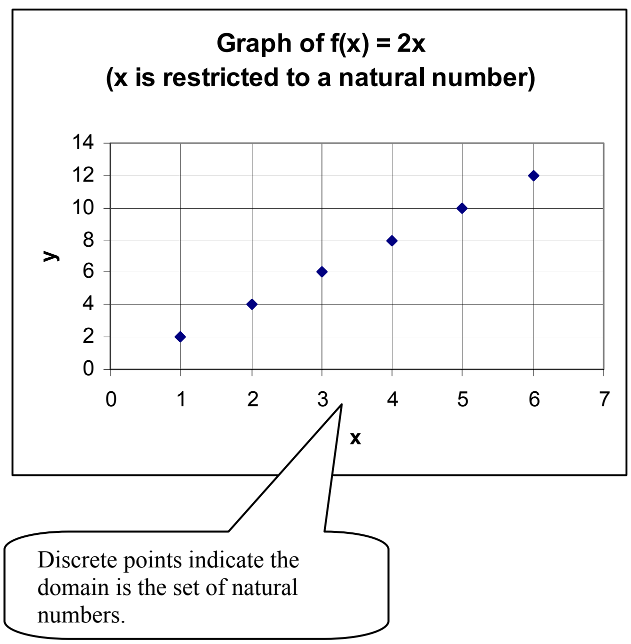
Example 23.3 – Determining the domain from an equation:

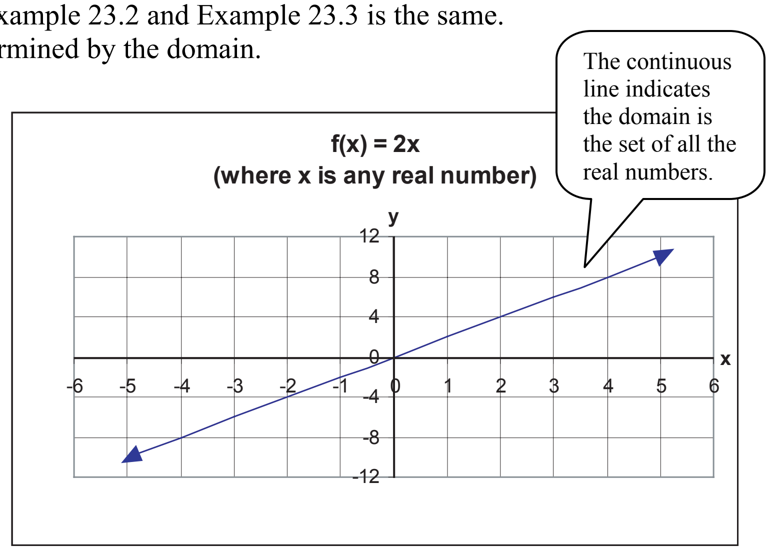
If an equation is given to represent a function and the domain is not explicitly indicated, it is assumed that the domain of the function is the set of all real numbers that are allowable for the independent variable.

$$f(x) = 2x \qquad (y = 2x)$$

The domain of *f* is the set of all real numbers since **ANY** real number is allowable for *x*.

Note: The relationship between *x* and *y* in Example 23.2 and Example 23.3 is the same. The difference between the functions is determined by the domain.





Example 23.4:

What is the domain of $f(x) = \dfrac{x+2}{x-3}$?

Answer: The domain is the set of allowable values for *x*. Since any real number except 3 (which causes division by zero) is allowable for *x*, the domain is the set of all real numbers except 3.

F&A – 24 **Range of a function:** The range of a function is the set consisting of all of the second coordinates of all the ordered pairs that are members of *f*. The range is dependent upon the domain of the function.

Example 24.1: $S = \{(-1, 3), (0, 4), (2, 7), (5, 8)\}$

The range of S is $\{3, 4, 7, 8\}$.

Example 24.2: Consider the function *f* consisting of all ordered pairs where the second coordinate is twice the first coordinate and the first coordinate is restricted to be a natural number.

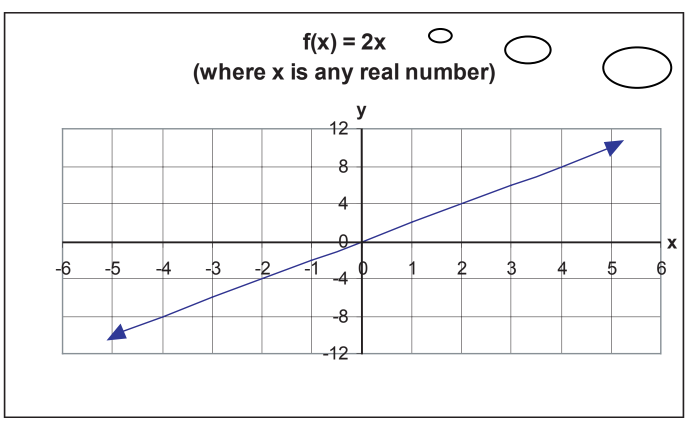
x is restricted to a natural number.

x	*y*
1	2
2	4
3	6
4	8
5	10
6	12
⋮	⋮

The range, in this case, is the set of even natural numbers or shown as {2, 4, 6, 8,...}.

Example 24.3: Using Example 23.3, what is the range of $f(x) = 2x \;\; (y = 2x)$?

The domain of *f* is the set of all real numbers (See Example 23.3). It can be seen that the range of *f* is the set of all real numbers, since any real number can be obtained by substituting half the value for *x* (e.g., the value of 6 can be obtained for *y* by substituting 3 for *x*).



Because the domain of $f(x) = 2x$ is the set of all real numbers, the range is the set of all the real numbers.

(Definition *F&A – 24* continued on following page)

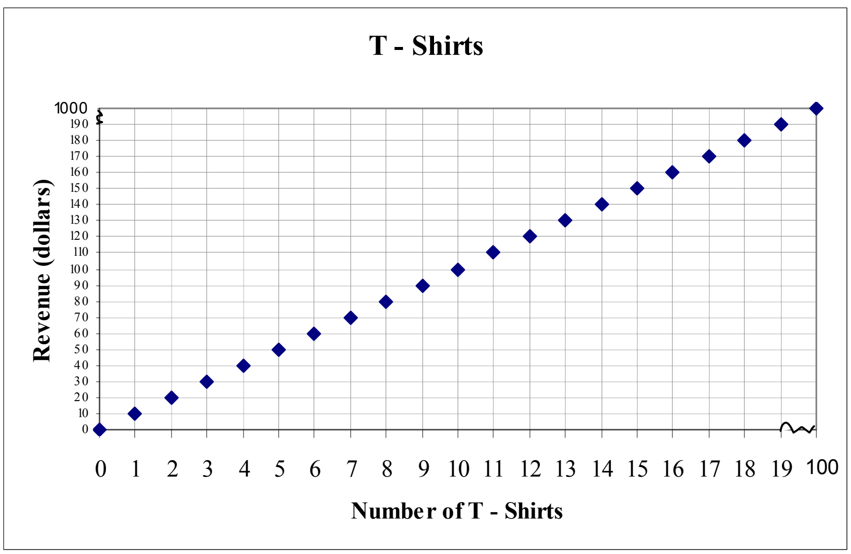
Example 24.4 – Considering domain and range:

A basketball team printed 100 t-shirts to celebrate their state championship. The team will sell the t-shirts at a cost of \$10.00 per shirt to raise money to attend summer basketball camp.

The function $R(x) = 10x$ represents the revenue (amount of money raised) for selling x t-shirts. In this situation, the number of t-shirts is restricted to 100. Therefore, the revenue is restricted to

$$R(x) = 10 \cdot 100 = \$1000.$$

Additionally, it is not possible to sell fractional parts of t-shirts. Therefore, the domain of R is the set of all whole numbers from 0 to 100 and the range is the set of all whole numbers from 0 to 1000 that are multiples of 10.



Resource Material Prototype

Section 3: Algebraic expressions

NECAP M(F&A) – X – 3

Vermont GLE MX: 21

F&A – 25 **Algebraic expression:** An algebraic expression is a collection of numbers, variables, grouping symbols (e.g. parenthesis, brackets), and operators (e.g., addition signs, subtraction signs, multiplication signs, square roots, exponents).

Example 25.1 – Algebraic expression: $3x + 25 – 2(2y + 12)$

$3x + 25 – 2(2y + 12)$

Variables

$3x + 25 – 2(2y + 12)$

Grouping Symbols

3x means 3 multiplied by x.

$3x + 25 – 2(2y + 12)$

Operators

Example 25.2 – Algebraic expressions found in equations:

$$\overbrace{F^o}^{\text{Expression}} = \underbrace{\frac{9}{5}C^o + 32}_{\text{Expression}}$$

$$\overbrace{3(\sqrt{x} + 4)}^{\text{Expression}} = \underbrace{16 - x}_{\text{Expression}}$$

F&A – 26 **Evaluating algebraic expressions:** Evaluating an algebraic expression means to determine the result of the algebraic expression after the values of the variables are substituted into the equation.

Example 26.1: Evaluate $3x + 4y$, when $x = 5$ and $y = 3$.

$3x + 4y = 3(5) + 4(3)$	Substitute 5 for x and 3 for y.
$= 15 + 12$	Find the value of the resulting numeric expression.
$= 27$	

F&A – 27 **Simplifying algebraic expressions:** Simplifying an algebraic expression means to reduce the expression to a simpler form. Often, this is accomplished by combining like terms and applying appropriate properties and operations consistent with order of operation conventions. Though an algebraic expression can often be rewritten in several simpler forms, usually the goal is to write an algebraic expression in the simplest form.

Example 27.1: Simplify $3[3(x+25x)+2(\sqrt{36}-x)]$.

The process below illustrates a number of simpler forms of the expression $3[3(x+25x)+2(\sqrt{36}-x)]$ resulting from simplifying $3[3(x+25x)+2(\sqrt{36}-x)]$ by combining like terms, applying order of operation conventions, and properties. The final expression (in part f) represents the simplest form of the original expression.

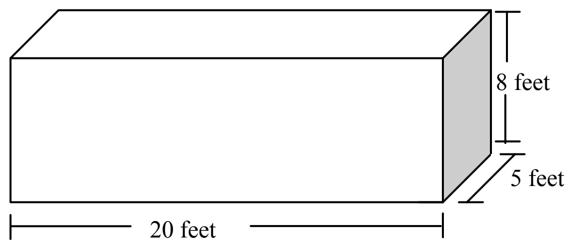
Algebraic Expression	Action Taken
a) $3[3(x+25x)+2(\sqrt{36}-x)]$ $3[3(26x)\quad +2(\sqrt{36}-x)]$	Combine like terms ($x + 25x = 26x$).
b) $3[3(26x)\quad +2(\sqrt{36}-x)]$ $3[3(26x)\quad +2(6-x)]$	Determine $\sqrt{36}$.
c) $3[3(26x)\quad +2(6-x)]$ $3[\ 78x\quad +2(6-x)]$	Multiply 3 and $26x$.
d) $3[\ 78x\quad +2(6-x)]$ $3[\ 78x\ +12-2x]$	Apply distributive property by multiplying each term inside the parenthesis by 2: ($2 \cdot 6 - 2 \cdot x = 12 - 2x$).
e) $3[\ 78x\ +12-2x]$ $3[\ 76x+12]$	Combine like terms: ($78x - 2x = 76x$).
f) $3[\ 76x+12]$ $228x+36$	Apply distributive property by multiplying each term inside the brackets by 3: ($3 \cdot 76x + 3 \cdot 12 = 228x + 36$).

Note: "Simplest" is not tightly defined. Often the simplest form is determined by the problem situation or context in which the expression is being used (e.g., $3(76x + 12)$ may be considered a simpler form of $228x + 36$ if the goal is to express $228x + 36$ in factored form).

F&A – 28 **Formula:** A formula is a general rule (e.g., area, volume, or surface area of two and three-dimensional geometric figures, $d = rt$).

Example 28.1 – Using a known formula: The formula to determine the volume of a rectangular prism is $V = lwh$, where V represents volume, l represents length, w represents width, and h represents height.

Determine the volume of the rectangular prism.



$$V = lwh, \text{ where } l = 20 \text{ feet}, w = 5 \text{ feet, and } h = 8 \text{ feet}$$

Answer: $V = 800 \text{ ft}^3$

F&A – 29 **Write equivalent forms of formulas:** To write equivalent forms of formulas means to solve for different variables in terms of the other variables in the generalized formula.

Example 29.1 – Equivalent forms of a known formula: The formula to determine the volume of a rectangular prism is $V = lwh$, where V represents volume, l represents length, w represents width, and h represents height.

Equivalent forms of $V = lwh$:

$$\frac{V}{lw} = h \qquad \frac{V}{lh} = w \qquad \frac{V}{hw} = l$$

Resource Material Prototype

Section 4: Demonstrate understanding of equality

NECAP GLE: M(F&A) – X – 4
Vermont GLE: MX: 22

	Page #	Definition #
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Demonstrates equality	50	*F&A – 31*
Equality	50	*F&A – 30*
Equation	52	*F&A – 32*
Examples of forms of equations	54	*F&A – 35*
Number sentences	52	*F&A – 33*

Resource Material Prototype

Section 4: Demonstrate understanding of equality

F&A – 30 **Equality:** Equality refers to the condition in which two expressions that have the same value are joined by an equal sign. These two expressions are called equivalent expressions.

F&A – 31 **Demonstrates conceptual understanding of equality by showing equivalence:** To demonstrate conceptual understanding of equality by showing equivalence means to illustrate that two expressions joined by an equal sign have the same value. Each example below shows a way in which students will be asked to demonstrate understanding of equality by showing equivalence.

Example 31.1: Find the value that will make an open sentence true (Grade 1 and up)

$$2 + \square = 7$$
$$\square = 5$$

Example 31.2: Show that the expressions connected by the equal sign have the same value using models or other representations of the expressions (Grade 3 and up)

$$4 + 5 = 3 + 6$$
$$\blacksquare\blacksquare\blacksquare\blacksquare + \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare = \blacksquare\blacksquare\blacksquare + \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare$$
$$\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare = \blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare$$
$$9 = 9$$

Example 31.3: Use models to show equivalence (Grade 3 and up)
Each of these shapes represents a value.

$$\square \qquad \blacktriangle \qquad \bullet$$

What is the value for each shape in the number sentences below? Each shape stands for the same value no matter where it is placed.

$$\square + \square + \square = 12$$
$$\square + \bullet + \blacktriangle = 12$$
$$\blacktriangle + \blacktriangle + \blacktriangle + \blacktriangle = 12$$

Answer: $\blacktriangle = 3$; $\square = 4$; $\bullet = 5$

(Definition *F&A – 31* continued on following page)

Example 31.4: Solve linear equations (Grade 4 and up)

See forms and examples for each grade level in Table 38.1 (page 55).

Example 31.5: Determine which values from a replacement set make an equation true (Grade 5)

Given $\{x: x = 2, 3, 4, 5\}$ determine which value(s) for x make the following equation true.

Read as "The set of x such that x equals 2, 3, 4, or 5."

$$2x+3=11$$
$$2(2)+3\stackrel{?}{=}11$$
$$7\neq 11$$
$$2(3)+3\stackrel{?}{=}11$$
$$9\neq 11$$
$$2(4)+3\stackrel{?}{=}11$$
$$11=11 \quad \text{(true statement)}$$
$$2(5)+3\stackrel{?}{=}11$$
$$13\neq 11$$

The number 4 is the only value for x in the given set that makes the equation true.

Example 31.6: Translating a problem situation into an equation (Grade 7 and up)

You are buying two CDs that each cost $15.95 including tax. You have $20.00. How much more money do you need to purchase the CDs?

Write an equation that represents this situation.

Answer: 2($15.95) = $20.00 + x, where x is the additional money that you need to purchase the two CDs, is an example of an equation that represents this situation.

F&A – 32 **Equation:** An equation is a statement representing two equivalent expressions joined by an equal sign (=).

Table 32.1: The following table organizes examples of equations by showing their equivalent expressions.

Equation	Equivalent expressions	Type of expressions
$3 + 6 + 2 = 12 - 1$	$3 + 6 + 2$ and $12 - 1$	numeric
$21 = 3 \times 7$	21 and 3×7	numeric
$8 = 8$	8 and 8	numeric
$2x + 7 = 15$	$2x + 7$ and 15 (when $x = 4$)	algebraic and numeric
$3x + 5 = 24 - 2x$	$3x + 5$ and $24 - 2x$ (when $x = \frac{19}{5}$)	algebraic

F&A – 33 **Number sentences:** Equations and inequalities involving numbers or unknowns are number sentences. A number sentence is <u>sometimes</u> referenced by the type of operation within it.

Number sentence	Type
$3 + 4 = 7$	Addition sentence
$47 - 19 - 3 > 23$	Subtraction sentence
$6 \times 9.5 = 19 \times 3$	Multiplication sentence
$86 \div 43 < 5$	Division sentence
$3 + 4 = \square$	Addition sentence
$86 \div y = 2$	Division sentence

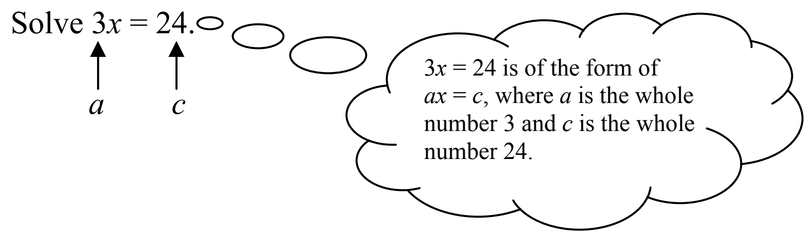
Note: When a number sentence contains more than one type of operation, it is <u>not</u> referenced by the combination of operations. For example, $6 \times 9 - 18 + 5 = 41$ is <u>not</u> referenced as a "multiplication-subtraction-addition" sentence.

At the lower elementary grades, a number sentence with an unknown is often referred to as an open number sentence (e.g., $3 + 4 = \square$). At the upper elementary grades, a number sentence with an unknown is often referred to as an equation or inequality (e.g., $86 \div y = 2$).

F&A – 34 **Algebraic equation notation used in GLEs:** The GLEs indicate the types of equations that will be assessed at a given grade level (e.g., In fourth grade, students are assessed on solving one-step linear equations of the form $ax = c$, $x \pm b = c$, where a, b, and c are whole numbers with $a \neq 0$.).

(See Table 35.1 [page 55] with parameters for the appropriate equations at different grade levels.)

Example 34.1: $ax = c$, where a and c are whole numbers and $a \neq 0$:



Answer: Since replacing x with 8 makes this open sentence produce a true statement, 8 is the solution for this equation.

Note: An equation of the form $\boldsymbol{ax = c}$ can also be written in the form $\boldsymbol{c = ax}$.

$\boldsymbol{ax = c}$	$\boldsymbol{c = ax}$
$3x = 24$	$24 = 3x$
$x = 8$ (Note: here $a = 3$ and $c = 24$)	$8 = x$ (Note: here $a = 3$ and $c = 24$)

Example 34.2: $x \pm b = c$, where b and c are whole numbers:

The $\pm$ in the equation indicates that the equation can be stated either as $x + b = c$ **or** $x - b = c$.

$\boldsymbol{x + b = c}$	$\boldsymbol{x - b = c}$
$x + 5 = 20$	$x - 5 = 20$
Since replacing x with 15 makes this open sentence produce a true statement, 15 is the solution for this equation.	Since replacing x with 25 makes this open sentence produce a true statement, 25 is the solution for this equation.

F&A – 35 Examples of forms of equations:

Table 35.1:

Examples of equations (at different grade levels) consistent with parameters set			
Types of Equations in GLE	**Grade**	**Examples**	**Solutions**
Equations of the form $ax = c$, where a and c are whole numbers with $a \neq 0$	4 and above	$3x = 15$ $15 = 3x$	Whole Numbers
Equations of the form $x \pm b = c$, where a, b, and c are whole numbers with $a \neq 0$	4 and above	$x + 3 = 12$ $12 = 3 + x$ $3 + x = 12$ $x - 5 = 15$ $15 = x - 5$	Whole numbers
Equations of the form $\frac{x}{a} = c$, where a and c are whole numbers with $a \neq 0$	5 and above	$\frac{x}{5} = 3$ $3 = \frac{x}{5}$ …or equivalent forms… $\frac{1}{5}x = 3$ $3 = \frac{1}{5}x$	Whole numbers
Multi-step linear equations of the form $ax \pm b = c$, where a, b, and c are whole numbers with $a \neq 0$	6 and above	$2x + 5 = 15$ $2x - 5 = 15$	Whole numbers, fractions, or decimals
Multi-step linear equations of the form $(x/a) \pm b = c$ with $a \neq 0$, where a, b, and c are whole numbers.	7 and above	$\frac{1x}{5} + 7 = 20$ …or equivalent forms… $\frac{1}{5}x + 7 = 20$	Whole numbers, fractions, or decimals
Multi-step linear equations of the form $ax \pm b = cx \pm d$ with $a, c \neq 0$, where a, b, c and d are whole numbers.	7 and above	$4x + 25 = 20x + 30$	Whole numbers, fractions, or decimals
Multi-step linear equations with integer coefficients	8 and above	$3x + 15 = 12 - 3x$ $-3x + 33 = -12x - 25$ $-3x + 7 = 20$	Whole numbers, fractions, decimals, or integers
Multi-step linear equations with rational coefficients	HS	$-\frac{3x}{5} + 15 = \frac{3}{4} + 12x$	Whole numbers, fractions, decimals, or integers

Appendix A: References

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